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NON-PARAMETRIC HYPERSURFACES WITH BOUNDED CURVATURES

HARLEY FLANDERS

1. Introduction

We work with the hypersurface in E^{n+1} which is the graph of a C'' function

$$u = u(x^1, \cdots, x^n)$$

on an open ball $\sum(x^i)^2 < R^2$ in E^n. We use the notation

$$p_i = \partial u/\partial x^i , \qquad r_{ij} = \partial^2 u/\partial x^i \partial x^j ,$$
$$p = (p_1, \cdots, p_n) , \qquad R = \|r_{ij}\| ,$$
$$w^2 = 1 + |p|^2 = 1 + \sum p_1^2 .$$

We also introduce the matrix

$$A = -\frac{1}{w} R\left(I - \frac{1}{w^2}\, {}^t pp\right) .$$

It is known that A has geometrical significance, indeed, its characteristic roots are the principal curvatures of the hypersurface (see Flanders [3, pp. 116–126] for details). The various curvatures K_1 = mean curvature, $K_2, \cdots, K_n$(= total curvature) are given by the characteristic polynomial:

$$\det(tI - A) = t^n - \binom{n}{1} K_1 t^{n-1} + \binom{n}{2} K_2 t^{n-2} - \cdots + (-1)^n K_n .$$

In [5], Heinz proved that if $|K_1| \geq a < 0$ for the function $u = u(x, y)$ of two variables defined over $x^2 + y^2 < R^2$, then $R \leq 1/a$. This was generalized to n variables in Chern [1, Theorem 1] and independently in Flanders [4]. Again the hypothesis $|K_1| \geq a > 0$ leads to $R \leq 1/a$ and this is best possible.

Heinz [5] also considered a surface $u = u(x, y)$, $x^2 + y^2 < R^2$, for which the total (Gaussian) curvature

$$K_2 = \frac{rt - s^2}{w^4} , \qquad (w^2 = 1 + p^2 + p^2)$$

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is bounded away from zero. His results are different for different sign. Precisely he proved (i) if $K_2 \geq a > 0$, then $R \leq 1/\sqrt{a}$ and (ii) if $K_2 \leq -a < 0$, then $R \leq e\sqrt{3}/\sqrt{a}$. The first of these results is an easy consequence of the situation in which K_1 is bounded away from zero and we shall give the n-variable form of this result in the next section. The second result is more difficult and depends on the following integral formula due to S. Bernstein:

$$(1.1)\qquad \begin{aligned} 2 \iint_{x^2+y^2 \leq R^2} (rt - s^2)dxdy \\ = \int_0^{2\pi} \left(\frac{\partial u}{\partial r}\right)^2 d\theta - \frac{d}{dr}\left[\frac{1}{r}\int_0^{2\pi}\left(\frac{\partial u}{\partial \theta}\right)^2 d\theta\right]_R . \end{aligned}$$

Chern [1, Formula 71] gives the generalization of this formula which is the first step for extending Heinz's proof to more than two variables. Using this Chern [1, Theorem 4] shows that if $K_2 \leq -a < 0$ and another rather complicated inequality is satisfied, then R is bounded. It does not seem likely that this technique of proof is adequate for obtaining a bound on R from the single hypothesis that $K_2 \leq -a$, indeed such a result may not exist. It is possible however to use the method of proof for results in a different direction which may turn out to be more natural. Instead of the curvatures we use the invariants of $R = \|r_{ij}\|$ divided by suitable powers of w. Since results for the first and second of these work, one may conjecture that such are true for all of the invariants. In particular one may conjecture that for the last of these

$$K_n = \frac{\pm |R|}{w^{n+2}} ,$$

an inequality $|K_n| \geq a > 0$ bounds the domain.

2. $K_2 \geq a > 0$

Theorem 1. *Let $u = u(x^1, \cdots, x^n)$ be defined on $|x| < R$ and suppose $K_2 \geq a > 0$. Then $R \leq 1/\sqrt{a}$.*

Proof. We prove first that $K^2 \leq K_1^2$. At a given point x let $\lambda_1, \cdots, \lambda_n$ be the principal curvatures. Then

$$nK_1 = \sum \lambda_i , \qquad n(n-1)K_2 = \sum_{i \neq j} \lambda_i \lambda_j ,$$
$$n^2 K_1^2 = \sum \lambda_i^2 + \sum_{i \neq j} \lambda_i \lambda_j .$$

For each i, j we have $\lambda_i^2 + \lambda_j^2 \geq 2\lambda_i\lambda_j$; hence by summing,

$$(n-1)\sum \lambda_i^2 \geq \sum_{i\neq j} \lambda_i\lambda_j ,$$

$$n^2(n-1)K_1^2 = (n-1)\sum \lambda_i^2 + (n-1)\sum_{i\neq j}\lambda_i\lambda_j$$

$$\geq n \sum_{i\neq j} \lambda_i\lambda_j = n^2(n-1)K_2 ,$$

and $K_1^2 \geq K_2$ follows. Thus $K_1 \geq \sqrt{a}$; so by Chern [1, Theorem 1] or Flanders [4] we have $R \leq 1/\sqrt{a}$.

Remark. Chern gave a slightly weaker result in [1, Theorem 2] and the present result in the Corollary to Theorem 1 of [2]. Our proof is different.

3. The Bernstein-Chern formula

In this section we state in our notation Chern's generalization [1, Formula 71] of (1.1).

Let $u = u(x^1, \cdots, x^n)$ be a C'' function on $|x| < R$ and let $0 < r < R$. Then we have

$$\tag{3.1} \begin{aligned} &2 \int_{|x|\leq r} \left(\sum_{i<j} \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix}\right) dx^1 \cdots dx^n \\ &= r^{n-2}\left[(n-1)\int_{|x|=r}\left(\frac{\partial u}{\partial r}\right)^2 \sigma - \frac{d}{dr}\left(r\int_{|x|=r} |\operatorname{grad} u|^2\sigma\right)\right] . \end{aligned}$$

The notation is the following. The form σ is the element of volume on the unit sphere S^{n-1}. At $|x| = r$, the ordinary space gradient $\nabla u = p = (p_1, \cdots, p_n)$ has a radial component of length $\partial u/\partial r$ and a component tangential to rS^{n-1}. It is this which we denote grad u so that

$$\nabla u = p = \frac{\partial u}{\partial r}\frac{x}{r} + \operatorname{grad} u .$$

We shall discuss an alternate derivation of (3.1) elsewhere.

4. Bounded trace

We shall replace the matrix A by the matrix $R = \|r_{ij}\|$ divided by suitable powers of w. We have the following result.

Theorem 2. *Let $u = u(x)$ be a C'' function on $|x| < R$ which satisfies*

$$\left|\frac{1}{nw^{1+1/n}}\sum r_{ii}\right| \geq a > 0 .$$

Then

$$R \leq \frac{e}{a}\,.$$

Proof. For each $x \in rS^{n-1}$, set $y = x/r \in S^{n-1}$. Then y is the unit normal at x to rS^{n-1}. We recall the standard formula for the vector area element $r^{n-1}\sigma y$ of rS^{n-1}:

$$r^{n-1}\sigma y = (dx^2 \cdots dx^n, \, -dx^1dx^3 \cdots dx^n, \cdots) = *dx\,,$$

where $*$ is the usual adjoint in E^n (Flanders [3, pp. 136 ff.]).

We use $du = p \cdot dx$, $*du = p \cdot *dx$, and $d*du = (\sum r_{ii})dx^1 \cdots dx^n$ to compute:

$$\begin{aligned}\int_{|x|\leq r} (\textstyle\sum r_{ii})dx^1 \cdots dx^n &= \int_{|x|\leq r} d*du = \int_{|x|=r} *du \\ &= \int_{|x|=r} p \cdot *dx = r^{n-1}\int_{|x|=r} p \cdot y\sigma \\ &\leq r^{n-1}\int_{|x|=r} |p|\sigma < r^{n-1}\int_{|x|=r} w\sigma;\end{aligned}$$

hence

$$\int_{|x|\leq r} (\textstyle\sum r_{ii})dx^1 \cdots dx^n < r^{n-1}\int_{|x|=r} w\sigma\,. \tag{4.1}$$

(This is the analogue of (3.1) for the first invariant of R. Most desirable would be a corresponding relation for the general invariant.)

We set

$$b = |S^{n-1}| = \int_{S^{n-1}} \sigma$$

so that

$$\int_{|x|\leq r} dx^1 \cdots dx^n = \int_0^r t^{n-1}dt \int_{S^{n-1}} \sigma = \frac{b}{n}\,r^n\,.$$

Replacing u by $-u$ we see that there is no loss of generality in supposing $\sum r_{ii} > 0$, which we do.

We now define

$$f(r) = \int_{|x|\leq r} w\,dx^1 \cdots dx^n = \int_0^r t^{n-1}dt \int_{|x|=r} w\sigma\,,$$

and assert

(i) $$f(r) \geq \frac{br^n}{n} ,$$

(ii) $$\int_{|x|\leq r} w^{1+1/n} dx^1 \cdots dx^n \geq \left(\frac{n}{b}\right)^{1/n} \frac{1}{r} f(r)^{1+1/n} ,$$

(iii) $$f'(r) \geq na\left(\frac{n}{b}\right)^{1/n} \frac{1}{r} f(r)^{1+1/n} .$$

The first of these is true because $w \geq 1$; the second follows by application of the Hölder inequality:

$$f(r) = \int_{|x|\leq r} w dx^1 \cdots dx^n \leq \left(\int w^{1+1/n}\right)^{n/(n+1)} \left(\int 1\right)^{1/(n+1)} .$$

For (iii) we have

$$f'(r) = r^{n-1} \int_{|x|=r} w\sigma \geq \int_{|x|\leq r} (\textstyle\sum r_{ii})\, dx^1 \cdots dx^n$$

$$\geq na \int_{|x|\leq r} w^{1+1/n} dx^1 \cdots dx^n ,$$

and (ii) applies. We thus have

$$\frac{f'(r)}{f(r)^{1+1/n}} \geq na\left(\frac{n}{b}\right)^{1/n} \frac{1}{r} .$$

We let $0 < r < r_1 < R$ and integrate:

$$n\left[\frac{1}{f(r)^{1/n}} - \frac{1}{f(r_1)^{1/n}}\right] \geq na\left(\frac{n}{b}\right)^{1/n} \ln (r_1/r) .$$

We drop the second term on the left and let $r_1 \to R$:

$$\frac{1}{f(r)^{1/n}} \geq a\left(\frac{n}{b}\right)^{1/n} \ln (R/r) .$$

By this and (i),

$$\frac{1}{r} \geq a \ln(R/r) .$$

The choice $r = R/e$ is the best and yields the conclusion.

We may now duplicate the proof of Theorem 1 as applied to the symmetric matrix $R/w^{1+1/n}$ instead of A to derive the following theorem.

Theorem 3. *Let $u = u(x)$ be a C'' function on $|x| < R$ and suppose*

$$\frac{1}{\binom{n}{2} w^{2+2/n}} \sum_{i<j} \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix} \geq a > 0 .$$

Then

$$R \leq \frac{e}{\sqrt{a}} .$$

We note that if the exponent of w in the hypothesis of Theorem 2 is increased then the resulting bound on R may be decreased. The proof of Theorem 2 works with this modification: the final step involves a power of r rather than the logarithm. This logarithm seems to be the critical case. It appears in Heinz's [5] proof of his Theorem 4 and again in our proof below of Theorem 6. (It is not this critical case which appears in Chern's [1, Theorem 4].)

Theorem 4. *Let $u = u(x)$ be a C'' function on $|x| < R$ and suppose*

$$\frac{1}{n w^{1+\delta/n}} \left| \sum r_{ii} \right| \geq a > 0 \qquad (\delta > 1) .$$

Then

$$R \leq \frac{\delta^{1/(\delta-1)}}{a} .$$

Proof. We follow the proof of Theorem 2 closely. As before we assume $\sum r_{ii} > 0$ and set

$$f(r) = \int_{|x| \leq r} w dx^1 \cdots dx^n .$$

In order, we derive

(i) $$f(r) \geq \frac{br^n}{n} \qquad (b = |S^{n-1}) ,$$

(ii) $$\int_{|x| \leq r} w^{1+\delta/n} \geq \left(\frac{n}{b}\right)^{\delta/n} \frac{1}{r^\delta} f^{1+\delta/n} ,$$

(iii) $$f'(r) \geq na \left(\frac{n}{b}\right)^{\delta/n} \frac{1}{r^\delta} f(r)^{1+\delta/n} .$$

We integrate (iii) from r to r_1, $0 < r < r_1 < R$:

$$\frac{n}{\delta}\left(\frac{1}{f(r)^{\delta/n}} - \frac{1}{f(r_1)^{\delta/n}}\right) \geq \left[na\left(\frac{n}{b}\right)^{\delta/n}\right]\left(\frac{1}{\delta - 1}\right)\left(\frac{1}{r^{\delta-1}} - \frac{1}{r_1^{\delta-1}}\right).$$

We drop the second term on the left, let $r_1 \to R$ and use (i) to derive

$$\frac{\delta - 1}{\delta a} \geq r - \frac{r^{\delta}}{R^{\delta-1}}.$$

The right hand side takes its max at $r^{\delta-1} = R^{\delta-1}/\delta$. We substitute this value of r to obtain the conclusion of the theorem.

Having this result we may exploit the inequality on the second invariant of the symmetric matric $R/w^{1+\delta/n}$ to deduce the following.

Theorem 5. *Let $u = u(x)$ be a C'' function on $|x| < R$ and suppose*

$$\frac{1}{\binom{n}{2} w^{2+2\delta/n}} \sum_{i<j} \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix} \geq a > 0 \qquad (\delta > 1).$$

Then

$$R \leq \delta^{1/(\delta-1)}/\sqrt{a}.$$

5. Bounded negative second invariant

Our main result is the following.

Theorem 6. *Let $u = u(x)$ be C'' on $|x| < R$ in E^n and suppose*

$$\frac{1}{w^{2+4/n}} \sum_{i<j} \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix} \leq -a < 0.$$

Then

$$R \leq \left(e\sqrt{\frac{n(n+1)}{2}}\right)\frac{1}{\sqrt{a}}.$$

Proof. For $0 \leq r < R$, set

$$f(r) = \int_{|x|\leq r} (1 + |\operatorname{grad} u|^2)dx^1 \cdots dx^n$$
$$= \int_0^r t^{n-1}dt \int_{|x|=r} (1 + |\operatorname{grad} u|^2)\sigma,$$

where grad u is the tangential component of ∇u discussed in § 3. The function f has the following properties:

(i) $$f(r) \geq \frac{b}{n} r^n \qquad (b = |S^{n-1}|) ,$$

(ii) $$\int_{|x| \leq r} w^{2+4/n} dx^1 \cdots dx^n \geq \left(\frac{n}{b}\right)^{2/n} \frac{1}{r^2} f(r)^{1+2/n} ,$$

(iii) $$f'(r) = br^{n-1} + r^{n-1} \int_{|x|=r} |\operatorname{grad} u|^2 \sigma ,$$

(iv) $$f''(r) = br^{n-2} + \frac{n-2}{r} f'(r) + r^{n-2}(r \int_{|x|=r} |\operatorname{grad} u|^2 \sigma) .$$

Property (i) is immediate from the definition. To obtain (ii) we use

$$p = \nabla u = \frac{\partial u}{\partial r} \frac{x}{|x|} \operatorname{grad} u ,$$

$$w^2 = 1 + |p|^2 = 1 + \left(\frac{\partial u}{\partial r}\right)^2 + |\operatorname{grad} u|^2 ,$$

$$1 + |\operatorname{grad} u|^2 \leq w_2 .$$

Thus

$$f(r) \leq \int_{|x| \leq r} w^2 dx^1 \cdots dx^n \leq \left(\int (w^2)^{1+2/n}\right)^{n/(n+2)} \left(\int 1\right)^{2/(n+2)}$$

by Hölder's inequality, and so

$$f(r)^{(n+2)/n} \leq \left(\int_{|x| \leq r} w^{2+4/n} dx^1 \cdots dx^n\right) \left(\frac{b}{n} r^n\right)^{2/n} ,$$

and (ii) follows. Property (iii) follows from the definition of f. Having it we write the second r^{n-1} as $r^{n-2}r$ and differentiate:

$$\begin{aligned} f''(r) &= (n-1)br^{n-2} + (n-2)r^{n-2} \int (\cdots) + r^{n-2} \frac{d}{dr}\left(r \int (\cdots)\right) \\ &= (n-1)br^{n-2} + \frac{n-2}{r} (f' - br^{n-1}) + r^{n-2} \frac{d}{dr}\left(r \int (\cdots)\right) . \end{aligned}$$

Formula (iv) follows.

We now derive the differential inequality

$$f''(r) \geq \frac{n-2}{r} f'(r) + \frac{c}{r^2} f(r)^{1+2/n} ,$$

where

$$c = 2a\left(\frac{n}{b}\right)^{2/n}.$$

To do this we use, in order, (iv) above, (3.1), and the hypothesis of the theorem:

$$\begin{aligned} f''(r) - \frac{n-2}{r} f'(r) & \\ &\geq r^{n-2}\frac{d}{dr}\left(r\int_{|x|=r} |\operatorname{grad} u|^2\sigma\right) \\ &\geq -2\int_{|x|=r} \sum_{i<j}(r_{ii}r_{jj} - r_{ij}^2)dx^1 \cdots dx^n \\ &\geq 2a\int_{|x|\leq r} w^{2+4/n}dx^1 \cdots dx^n \\ &\geq 2a\left(\frac{n}{b}\right)^{2/n}\frac{1}{r^2} f(r)^{1+2/n}. \end{aligned}$$

We proceed to integrate this inequality. We have

$$\begin{aligned} \left(\frac{f'(r)^2}{r^{2(n-2)}}\right)' \geq \frac{2f'}{r^{2(n-2)}}\left(\frac{n-2}{r}f' + \frac{c}{r^2}f^{1+2/n}\right) & \\ - 2(n-2)\frac{f'^2}{r^{2(n-2)+1}} = \frac{2c}{r^{2(n-1)}} f^{1+2/n}f', \end{aligned}$$

and

$$\begin{aligned} \left(\frac{f^{2+2/n}}{r^{2(n-1)}}\right)' &= \left(2 + \frac{2}{n}\right)\frac{f^{1+2/n}f'}{r^{2(n-1)}} - 2(n-1)\frac{f^{2+2/n}}{r^{2(n-1)+1}} \\ &\leq \left(2 + \frac{2}{n}\right)\frac{f^{1+2/n}f'}{r^{2(n-1)}}, \end{aligned}$$

hence

$$\frac{f'(r)^2}{r^{2(n-2)}} \geq \frac{cn}{1+n}\left(\frac{f^{2+2/n}}{r^{2(n-1)}}\right)'.$$

As $r \to 0+$, $f(r) = O(r^n)$ so there is no problem integrating from 0 to r:

$$\frac{f'(r)^2}{r^{2(n-2)}} \geq \frac{cn}{1+n}\frac{f^{2+2/n}}{r^{2(n-1)}},$$

$$\frac{f'(r)}{f(r)^{1+1/n}} \geq \left(\frac{cn}{1+n}\right)^{1/2}\frac{1}{r}.$$

We let $0 < r_0 < r_1 < R$ and integrate again:

$$n\left(\frac{1}{f(r_0)^{1/n}}-\frac{1}{f(r_1)^{1/n}}\right)\geq\left(\frac{cn}{1+n}\right)^{1/2}\ln\left(\frac{r_1}{r_0}\right),$$

$$\frac{1}{f(r_0)^{1/n}}\geq\left(\frac{c}{n(1+n)}\right)^{1/2}\ln\left(\frac{r_1}{r_2}\right).$$

We may let $r_1\to R$, replace r_0 by r, and use (i) above to obtain

$$\left(\frac{n}{br^n}\right)^{1/n}\geq\left(\frac{c}{n(1+n)}\right)^{1/2}\ln\left(\frac{R}{r}\right),$$

$$\frac{1}{r}\geq\left(\frac{2a}{n(n+1)}\right)^{1/2}\ln\left(\frac{R}{r}\right).$$

This is true whenever $0<r<R$. To make the optimal choice of r, we set $r=vR$, $0<v<1$ and have

$$\frac{1}{R}\geq\left(\frac{2a}{n(n+1)}\right)^{1/2}(-v\ln v).$$

The maximal value of $(-v\ln v)$ is e^{-1} taken at $v=e^{-1}$; so the theorem follows.

In the conclusion of the theorem the fact that R is bounded by a constant over $\sqrt{a}$ is probably the best result. The particular constant we have, based as it is on some drastic estimates, is probably not the best. One may ask whether increasing the exponent on w in the hypothesis leads to an order of magnitude on R less than $\sqrt{a}$. The present method of proof leads only to a better constant. We have the following result.

Theorem 7. *Let u be a C'' function on $|x|<R$ in E^n and suppose*

$$\frac{1}{w^{2+4\delta/n}}\sum_{i<j}(r_{ii}r_{jj}-r_{ij}^2)\leq -a<0,$$

where $\delta>1$. Then

$$R\leqslant \delta^{1/(\delta-1)}\left(\frac{n(n+\delta)}{2}\right)^{1/2}\frac{1}{\sqrt{a}}.$$

The proof parallels the proof of Theorem 6 with a slight difference only at the last integration. We sketch the main points. The relation (ii) becomes

$$\int_{|x|\leq r} w^{2+4\delta/n}dx_1\cdots dx_n\geq\left(\frac{n}{b}\right)^{2\delta/n}\frac{1}{r^{2\delta}}f(r)^{(n+2\delta)/n}.$$

The differential inequality is replaced by

$$f''(r) \geq \frac{n-2}{r} f'(r) + \frac{c}{r^{2\delta}} f(r)^{1+2\delta/n} ,$$
$$c = 2a\left(\frac{n}{b}\right)^{2\delta/n} .$$

from which follows

$$\left(\frac{f'(r)^2}{r^{2(n-2)}}\right)' \geq \frac{2c}{r^{2(n-2+\delta)}} f^{1+2\delta/n} f'$$
$$\geq \frac{cn}{n+\delta}\left(\frac{f^{2+2\delta/n}}{r^{2(n-2+\delta)}}\right)' .$$

One integration leads to

$$\frac{f'}{f^{1+\delta/n}} \geq \left(\frac{cn}{n+\delta}\right)^{1/2} \frac{1}{r^{\delta}} .$$

Since $\delta > 1$ a second integration, where $0 < r_0 < r_1 < R$, leads to

$$\frac{n}{\delta}\left[\frac{1}{f(r_0)^{\delta/n}} - \frac{1}{f(r_1)^{\delta/n}}\right] \geq \frac{1}{(\delta-1)}\left(\frac{cn}{n+\delta}\right)^{1/2}\left(\frac{1}{r_0^{\delta-1}} - \frac{1}{r_1^{\delta-1}}\right) ,$$
$$\frac{1}{f(r)^{\delta/n}} \geq \frac{\delta}{(\delta-1)}\left(\frac{c}{n(n+\delta)}\right)^{1/2}\left(\frac{1}{r^{\delta-1}} - \frac{1}{R^{\delta-1}}\right)$$

for $0 < r < R$. By the estimate on f in (i) above,

$$\left(\frac{n}{b}\right)^{\delta/n} \frac{1}{r^{\delta}} \geq \frac{\delta}{\delta-1}\left(\frac{c}{n(n+\delta)}\right)^{1/2}\left(\frac{1}{r^{\delta-1}} - \frac{1}{R^{\delta-1}}\right) ,$$
$$\frac{1}{r^{\delta}} \geq \frac{\delta}{\delta-1}\left(\frac{2a}{n(n+\delta)}\right)^{1/2}\left(\frac{1}{r^{\delta-1}} - \frac{1}{R^{\delta-1}}\right) .$$

Setting $r = vR$, $0 < v < 1$ we find

$$\frac{1}{R} \geq \frac{\delta}{\delta-1}\left(\frac{2a}{n(n+\delta)}\right)^{1/2}(v - v^{\delta}) .$$

We maximize $v - v^{\delta}$ to complete the proof.

We remark that

$$\lim_{\delta\to 1+}\left[\delta^{1/(\delta-1)}\left(\frac{n(n+\delta)}{2}\right)^{1/2}\right] = e\left(\frac{n(n+1)}{2}\right)^{1/2} ,$$

the constant in Theorem 6. Indeed, it is easy to obtain Theorem 6 as a limiting case of Theorem 7 by the simple device of slightly decreasing R and a.

We note as a corollary the special case $\delta = n/2$.

Corollary 1. *Let u be a C'' function on* $|\boldsymbol{x}| < R$ *and suppose*

$$\frac{1}{w^4} \sum_{i<j} (r_{ii} r_{jj} - r_{ij}^2) \leq -a < 0 .$$

Then

$$R \leq \left(\frac{n}{2}\right)^{n/(n-2)} \frac{\sqrt{3}}{\sqrt{a}} .$$

7. Relation to curvature

We return to the considerations of § 7 and compute to what extent the expression in this corollary is related to the second curvature K_2.

We have

$$A = -\frac{1}{w} \left\| r_{ik} - \frac{1}{w^2} s_i p_k \right\| ,$$

where

$$s_i = \sum r_{ij} p_j .$$

The 2×2 principal minor of A is

$$\frac{1}{w^2} \begin{vmatrix} r_{ii} - \frac{1}{w^2} s_i p_i & r_{ik} - \frac{1}{w^2} s_i p_k \\ r_{ki} - \frac{1}{w^2} s_k p_i & r_{kk} - \frac{1}{w^2} s_k p_k \end{vmatrix}$$

$$= \frac{1}{w^2} \left[\begin{vmatrix} r_{ii} & r_{ik} \\ r_{ki} & r_{kk} \end{vmatrix} - \frac{s_k}{w^2} (p_k r_{ii} - p_i r_{ik}) - \frac{s_i}{w^2} (p_i r_{kk} - p_k r_{ki}) \right] .$$

Summing we find

$$\binom{n}{2} K_2 = \operatorname{tr} (\wedge^2 A)$$

$$= \frac{1}{w^4} [w^2 \operatorname{tr} (\wedge^2 R) - (\operatorname{tr} R)(pR\, {}^t p) + (pR^2\, {}^t p)] .$$

When $n = 2$ this formula reduces to the standard one for Gaussian curvature:

$$K = \frac{1}{w^4} \operatorname{tr} (\wedge^2 R) .$$

For $n > 2$ we can at least get some insight into the situation by assuming at some point that R is diagonal,

$$R = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} .$$

After a short calculation we find

$$\binom{n}{2} K_2 = \frac{1}{w^4} [\operatorname{tr}(\wedge^2 R) + \sum_{k=1}^{n} (\sum_{\substack{i<j \\ i \neq k \\ j \neq k}} \lambda_i \lambda_j) p_k^2] .$$

This shows that it is hopeless to expect that a bound on $[\operatorname{tr}(\wedge^2 R)]/w^4$ is implied by a bound on K_2 or visa-versa. It remains an open question whether an inequality of the form $K_2 \leq -a < 0$ forces a bound on R. Of course one may ask similar questions about $K_3, K_4, \cdots$ with both positive and negative bounds.

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